ACADEMIC INN INTERNATIONAL

ISSN: 2663-9513 (Online) ISSN: 2663-9505 (Print)

South Asian Journal of BIOLOGICAL RESEARCH

open access

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To cite the article: *Mst. Shahana Parvin[1*], Mst. Shamima Sultana[2]* (2024). *LAGRANGIAN FORMULATION OF NAVIER-STOKES EQUATIONS OF MOTIONS FOR THE SURFACE WAVES IN TWO-LAYERED FLUIDS, 6(1):1-14.*

Link to this article: **http://aiipub.com/journals/sajbr-240510-10014/**

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LAGRANGIAN FORMULATION OF NAVIER-STOKES EQUATIONS OF MOTIONS FOR THE SURFACE WAVES IN TWO-LAYERED FLUIDS

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ARTICLE INFO

Article Type: Research
Received: 08, May. 2024.
Accepted: 19, May. 2024.
Published: 29, May. 2024.

KEYWORDS:
Navier Stokes equations, Lagrangian formulation.

ABSTRACT

Depending on the specific situations, there are various types of equations can be employed to describe fluid flows. There are two kinds for describing the physical governing equations, the Navier-Stokes equations and Lagrangian equation. The Navier-Stokes description is a spatial description. On the other hand, Lagrangian description is a material description. The main focus is to highlight some important points for solving Navier-Stokes equations and its applications on different branches of fluid mechanics. Lagrangian formulations of Navier-Stokes equations for two layers of different densities subject to gravity force for irrotational, incompressible fluids are formulated.

1. INTRODUCTION

Navier-Stokes equations are not only used by mathematical field, but also seen outside of mathematics, such as physics, mechanics, ordinary fluid dynamics, magneto-hydrodynamics, water waves, meteorology, plasma physics, engineering fields etc. The Navier–Stokes equations are of great interest in a mathematical sense. The Navier- Stokes equations are a set of partial differential equations for describing the motion of viscous fluid substances according to French engineer and physicist Cloud-Louis Navier and Anglo-Irish physicist and mathematician George Gabriel Stokes. The Navier-Stokes equations mathematically express conservation of momentum and conservation of mass for Newtonian fluids. The most general form of the Navier-Stokes equations which cannot be applied until it has been more specified. An equation defining conservation of energy, Hamiltonian formulations are also constructed. Solutions to the **Navier–Stokes** equations are used in many practical applications. Navier-Stokes equations can be used to generalize the velocity vector field that applies to a fluid using some initial conditions and are taken from the application of Newton’s second law in combination with a fluid stress with viscosity and a pressure term.

Bona et al. (2002) derived four parameter family of Boussinesq systems from the two-dimensional Euler equation for small amplitude long waves in a nonlinear dispersive media for free surface flow and formulate criteria to help decide which of these equations might choose in a given modeling situation. Marche (2007) studied the derivation with asymptotic analysis of two-dimensional viscous shallow water model in rotating framework with irregular topography, linear and quadratic bottom terms and capillary effects considering the three-dimensional Navier-Stokes equations with a free moving surface boundary condition and hydrostatic approximation. Bona et al. (2005) obtained new nonlinear systems describing the interaction of long water waves in both two and three dimensions. Duchene (2010) derived asymptotic models for the propagation of two- and three-dimensional gravity waves at the free surface and the interface between two layers of immiscible fluids of different densities over an uneven bottom. Chen et al. (2010) investigated a water wave model with a nonlocal viscous term and the decay rate of solutions theoretically and numerically. Dean and Dalrymple (1991) derived wave energy and power which are nonlinear quantities such as time averaged, obtained from the linear wave theory. Longuet-Higgins (1987) showed a simple physical model to obtain the Lagrangian characters including particle motion, mass transport, the Lagrangian wave period and the Lagrangian mean level for the surface waves that cannot directly obtain throughout the entire flow field. Shamima Sultana and Zillur Rahman (2013) derived Hamiltonian formulation using Hamilton's principles for water wave evolution and also derived Hamilton's canonical equation of motion. Vikas et al. (2011) presented a brief review on Lagrangian-Hamiltonian mechanics and had a simple and new fundamental view on Lagrangian mechanics and was applied to investigate the dynamics of asymmetric and continuous systems. Cardoso-Ribeiro et al. (2020) extended previous work on shallow water equations using the port-Hamiltonian formulation by considering the two dimensional under rigid body motions. Perez and C. Ramirez (2015) discussed Lagrangian formulation for permionic and supersymmetric non-conservative systems. Parvin et al. (2014) derived mass transport velocity using boundary conditions at water bottom and the flap surface and established the time average of horizontal velocity component at zero mean level. Parvin et al. (2018) formulated the wave frequency in terms of wave number depending on linearized term and also established phase speed with viscous term for long wave length. Ambrosi (2000) established Hamiltonian formulation for surface waves in a layered fluid with gravity force. Haney et al. (2024) formulated incompressible Navier-Stokes equations which allow one to model boundary layer flows introduced by the motion of a deforming surface. Sadeghi (2023) derived Navier-Stokes equations from Lagrange's equations and also showed the derivation of governing equations for viscous flow the standard Lagrangian is more sufficient than non-standard Lagrangian. Yamazaki (2018) obtained stochastic Lagrangian formulations of solutions to some partial differential equations in fluid mechanics with diffusion, specifically damped Navier-Stokes equations, as well as the viscous and thermally diffusive Boussinesq system and then proved the global well-posedness of the four-dimensional Navier-Stokes equations with partial damping in only third and fourth components of the velocity field.

In this paper, a few concepts and mathematical tools for the foundation of fluid mechanics have been discussed. Fluid mechanics is a vast subject including diverse materials and phenomena. This paper shows some aspects of fluid mechanics that are relevant to the flow which gives solution of Navier-Stokes equations using variables separation method and also represents Lagrangian formulations for irrotational flow of two incompressible fluids of different densities with gravity force. Also, some graphs of Lagrangian formulations obtained from the solutions of Navier-Stokes equations generalized through the variable's separation method have been drawn with the help of symbolic computation softwar, like Maple-13.

2. MATHEMATICAL FORMULATIONS

We consider two fluids with constant densities ρ_1 and ρ_2, subject to the gravity force in a horizontally unbounded domain. The surface $z=\tau_1(x,y,t)=\tau_1(x,t)$ (the interface) separates the two fluids and the surface $z=\tau_2(x,y,t)=\tau_2(x,t)$ (the free surface) separates the upper fluid from the air. Suppose that the lower fluid is to be infinitely deep. At the far field the water is at rest, $\tau_1(\infty)=0$ and $\tau_2(\infty)=h=\text{constant}$. The velocity potentials of the lower and upper fluid are denoted by $\phi_1(x,z,t), \infty \le z \le \tau_1$ and $\phi_2(x,z,t), \tau_1 \le z \le \tau_2$ respectively. The motion of the fluid is also governed by Laplace equation.

The upper and lower surfaces are

$$F_1(x,z,t)=z-\tau_1(x,t)=0 \tag{2.1}$$

and

$$F_2(x,z,t)=z-\tau_2(x,t)=0 \tag{2.2}$$

The velocity of the fluids normal to the separating surface must be equal, therefore,

$$\frac{dF_1}{dt}=0 \text{ and } \frac{dF_2}{dt}=0 \tag{2.3}$$

Now

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial t}=u\frac{\partial z}{\partial x}+\frac{\partial z}{\partial t}$$

$$\frac{\partial \tau_1}{\partial t}=\frac{\partial \phi_1}{\partial z}-\nabla\tau_{1.}\nabla\phi_1 \quad for\ z=\tau_1$$

(2.4)

and

$$\frac{\partial \tau_2}{\partial t}=\frac{\partial \phi_2}{\partial z}-\nabla\tau_2.\nabla\phi_2 \quad for\ z=\tau_2 \tag{2.5}$$

The pressure gap across the interface and free surface is equal to the force per unit surface exerted by the surface tension.

$$P_1-P_2=-\tau_1\nabla.\frac{\nabla\tau_1}{\sqrt{1+(\nabla\tau_1)^2}}$$

$$P_2 = -\tau_2 \nabla . \frac{\nabla \tau_2}{\sqrt{1 + (\nabla \tau_2)^2}}$$

2.1 Variables Separation Method

Various methods are applied to solve Navier-Stokes equations in natural science. After a few simplifications of Navier-Stokes equations, velocity vector fields are derived using variables separation method. Though the potential energy of the above two systems is considered as infinite with respect to a reference state are neglected .

Let Ω be the domain which is occupied by viscous, incompressible fluid at time *t*. The system shows the Navier-Stokes equations of the motion in each fluid for upper and lower cases.

The Navier-Stokes equations of the motion is-

$$\rho\left(\frac{\partial \overline{V}}{\partial t} + (\overline{V}.\nabla)\overline{V}\right) = \overline{F} - \nabla p + \mu \nabla^2 \overline{V} \text{, in} \Omega \quad (2.1.1)$$

and for incompressible fluid

$$\nabla . \overline{V} = 0 \quad (2.1.2)$$

Equation (2.1.1) is also written as

$$\rho\left(\frac{\partial}{\partial t}(\nabla \phi) + (\nabla \phi . \nabla)(\nabla \phi)\right) = -\rho g \hat{k} - \nabla p + \mu \nabla^2 (\nabla \phi)$$

(2.1.3)

From equation (2.1.3), we can write for upper fluid of density ρ_1

$$\rho_1\left(\frac{\partial}{\partial t}(\nabla \phi_1) + (\nabla \phi_1 . \nabla)(\nabla \phi_1)\right) = -\rho_1 g \hat{k} - \nabla p_1 + \mu \nabla^2 (\nabla \phi_1)$$

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\nabla \phi_1)^2 = -gz - \frac{p_1}{\rho_1} + \upsilon \nabla^2 \phi_1, \; at \; z = \tau_1 \quad (2.1.4)$$

For small amplitude linear water waves, the Navier-Stokes equations of motion is linear. Then the flow of velocity potential $\Phi(x, z, t)$ is

$$\Phi(x, z, t) = \text{Re}\{\phi(x, z)e^{i\omega t}\}. \quad (2.1.5)$$

The following equations are hold

$$\phi_{xx} + \phi_{zz} = 0\,, -h < z < 0. \tag{2.1.6}$$

$$\left.\begin{aligned} \frac{\partial \phi}{\partial z} &= 0,\ z = h, \\ \frac{\partial \phi}{\partial z} &= K\phi,\ z = 0. \end{aligned}\right\} \tag{2.1.7}$$

The above equation can be obtained from combining the following two equations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial z} &= \pm i\omega\tau_1,\ z = 0 \\ \mu\, i\omega\phi &= g\tau_1,\ z = 0 \end{aligned}\right\} \tag{2.1.8}$$

where

$$\begin{aligned} K = \frac{\omega^2}{g} &= k\tanh(kh) \\ &= -k_n \tan(k_n h),\quad n = 1,2,\ldots\ldots\ldots\ldots \end{aligned} \tag{2.1.9}$$

Using variables separation method in equation (2.1.6), assuming

$$\phi_1(x,z) = F(x)G(z). \tag{2.1.10}$$

Therefore,

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dz^2}. \tag{2.1.11}$$

The L.H.S of equation (2.1.11) is only a function of x and the R.H.S of equation (2.1.11) is only a function of z, so each is constant. Consider

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dz^2} = \pm m^2. \text{ (Say)} \tag{2.1.12}$$

Taking positive sign, we have from equation (2.1.12),

$$\frac{d^2G}{dz^2} + m^2G = 0. \tag{2.1.13}$$

and

$$\frac{d^2F}{dx^2} - m^2F = 0. \tag{2.1.14}$$

Solving equation (2.1.13) we obtain

$G(z) = \alpha_1 e^{imz} + \beta_1 e^{-imz}$, where α_1 and β_1 are arbitrary constants.

Therefore, equation (2.1.10) becomes

$$\phi_1(x,z)=\left(\alpha_1 e^{imz}+\beta_1 e^{-imz}\right)F(x). \tag{2.1.15}$$

Applying boundary condition $\frac{\partial \phi_1}{\partial z}\Big|_{z=h}=0$ in equation (2.1.15), we have

$$\therefore \alpha_1=\beta_1 e^{-2imh}.$$

Therefore

$$G(z)=\gamma_1 \cos m(z-h). \tag{2.1.16}$$

where $\gamma_1 = 2\beta_1 e^{-imh} =$ constant.

Then equation (2.1.15) becomes

$$\phi_1(x,z)=\gamma_1 \cos m(z-h)F(x). \tag{2.1.17}$$

Again, solving equation (2.1.14) we have

$F=\alpha_2 e^{mx}+\beta_2 e^{-mx}$, where α_2 *and* β_2 are arbitrary constants.

For our convenience, if we set $\alpha_2=0$ *and* $\beta_2=1$, then $F=e^{-mx}$.

Therefore, equation (2.1.17) becomes

$$\phi_1(x,z)=\gamma_1 \cos m(z-h)e^{-mx}. \tag{2.1.18}$$

Applying boundary condition $\frac{\partial \phi_1}{\partial z}=\pm i\omega\tau_1,\ z=0$, we have

$$\frac{\partial \phi_1}{\partial z}=\gamma_1 m \sin(mh)e^{-mx}=i\omega\tau_1,\ z=0 \tag{2.1.19}$$

If we can write $\tau_1 = ae^{-mx}$, equation (2.1.19) becomes

$$\gamma_1=\frac{ai\omega}{m\sin(mh)}.$$

Hence

$$\phi_1(x,z)=\frac{ai\omega}{m\sin(mh)}\cos m(z-h)e^{-mx}. \tag{2.1.20}$$

and

$$G(z)=\frac{ai\omega}{m\sin(mh)}\cos m(z-h). \tag{2.1.21}$$

The imaginary solutions of this equation are denoted by $m_0 = \pm im$ and real solutions are also denoted by $m_n,\ for\ n \geq 1$. So we put $\omega^2 = gm\tanh(mh)$. Therefore, equation (2.1.9) can be written as

$$M = \frac{\omega^2}{g} = m\tanh(mh) \\ = -m_n\tan(m_n h), \quad n = 1,2,.........\,..... \tag{2.1.22}$$

Therefore, the function $G(z)$ is as

$$G(z) = \frac{a_n i\omega}{m_n \sin(m_n h)}\cos m_n(z-h),\ n \geq 0.$$

Also, we can write $F = e^{-m_n x}$.

Hence, the velocity potential expansion can be written as

$$\phi_1(x,z) = \frac{a_n i\omega}{m_n \sin(m_n h)}\cos m_n(z-h)e^{-m_n x}. \tag{2.1.23}$$

Using equation (2.1.22), we have

$$\phi_1(x,z) = -\frac{a_n i\omega}{M\cos(m_n h)}\cos m_n(z-h)e^{-m_n x}. \tag{2.1.24}$$

Therefore, equation (2.1.5) becomes

$$\Phi_1(x,z,t) = \mathrm{Re}\{\phi_1(x,z)e^{i\omega t}\}.$$

$$= \frac{a_n\omega}{M\cos(m_n h)}\cos m_n(z-h)\sin(\omega t)e^{-m_n x} \tag{2.1.25}$$

Similarly, we can write for lower fluid of density ρ_2, taking negative sign and applying the above boundary conditions which we already apply (positive sign) and taking for more convenient, the solution of equation (2.1.12) becomes

$$\phi_2(x,z) = -\frac{a_0\omega}{M\cosh(mh)}\cosh m(z-h)e^{-imx}.$$

(2.1.26)

Therefore, equation (2.1.5) becomes

$$\Phi_2(x,z,t) = \mathrm{Re}\{\phi_2(x,z)e^{i\omega t}\}.$$

$$= -\frac{a_0\omega}{M\cosh(mh)}\cosh m(z-h)\cos(mx-\omega t) \tag{2.1.27}$$

3. APPLICATIONS

A statement of upper fluid of density ρ_1 has been constructed in classical fluid dynamics when taking positive sign, Lagrangian formulation can be written as

$$L = \frac{1}{2}\rho_1\left(\left(\frac{\partial \Phi_1}{\partial x}\right)^2 + \left(\frac{\partial \Phi_1}{\partial z}\right)^2\right) - \int_{\tau_1}^{\infty} \rho_1 g dz \tag{3.1}$$

Using equation (2.1.25) in equation (3.1) and neglecting 2nd term, we have

$$L = Ae^{-2m_n x}\sin^2(\omega t) \tag{3.2}$$

where

$$A = \frac{1}{2}\rho_1 a_n^2 \omega^2 \operatorname{cos}ec^2(m_n h)$$

Similarly, a statement of lower fluid of density ρ_2 has been constructed in classical fluid dynamics when taking negative sign, Lagrangian formulation in sine or cosine form be in the following

$$L = \frac{1}{2}\rho_2\left(\left(\frac{\partial \Phi_2}{\partial x}\right)^2 + \left(\frac{\partial \Phi_2}{\partial z}\right)^2\right) - \int_{\tau_1}^{\tau_2} \rho_2 g dz \tag{3.3}$$

Using equation (2.1.25) in equation (3.3) , we have

$$\therefore L = \frac{1}{2}\rho_2 \frac{m^2 {a_0}^2 \omega^2}{M^2 \cosh^2(mh)}\left[\sin^2(mx-\omega t) + \sinh^2 m(z-h)\right]$$

Another form of Lagrangian function is

$$\Rightarrow L = B\left[\sin^2(mx-\omega t) + \sinh^2 m(z-h)\right] \tag{3.4}$$

or

$$L = B\left[\cosh^2 m(z-h) - \cos^2(mx-\omega t)\right] \tag{3.5}$$

where $B = \frac{1}{2}\rho_2 a_0^2 \omega^2 \operatorname{cos}ech^2(mh)$

4. THE GRAPHICAL REPRESENTATIONS AND EXPLANATIONS

Mathematically, a graph shows the diagram for the solutions of problems and describes obviously the character of the given system analytically or numerically or also expresses for closed-form solution of the system. Also it needs construction of the basic acquaintance of a graph signified and we have shown various types of Lagrangian formulations via this method. Therefore, some graphs of Lagrangian formulations from the obtained solutions of Navier-Stokes equations generalized in section-3 through the variables separation method have been drawn within the Figure-1-8 with the help of symbolic computation software, such as Maple-13.

5. CONCLUSIONS

The Navier- Stokes equations are governed by the surface wave in a layered fluid from which velocity potentials of upper and lower layers are derived. Velocity field can be derived from these two velocity potentials of the layers of two immiscible fluids. With the help of variables separation method and taking real part of these two velocity potentials we have had our desired result. Two fluids of different densities owing to gravitational body force, Lagrangian formulations of Navier-Stokes Equations for irrotational, incompressible fluid have been established which play a great significance in fluid dynamics.

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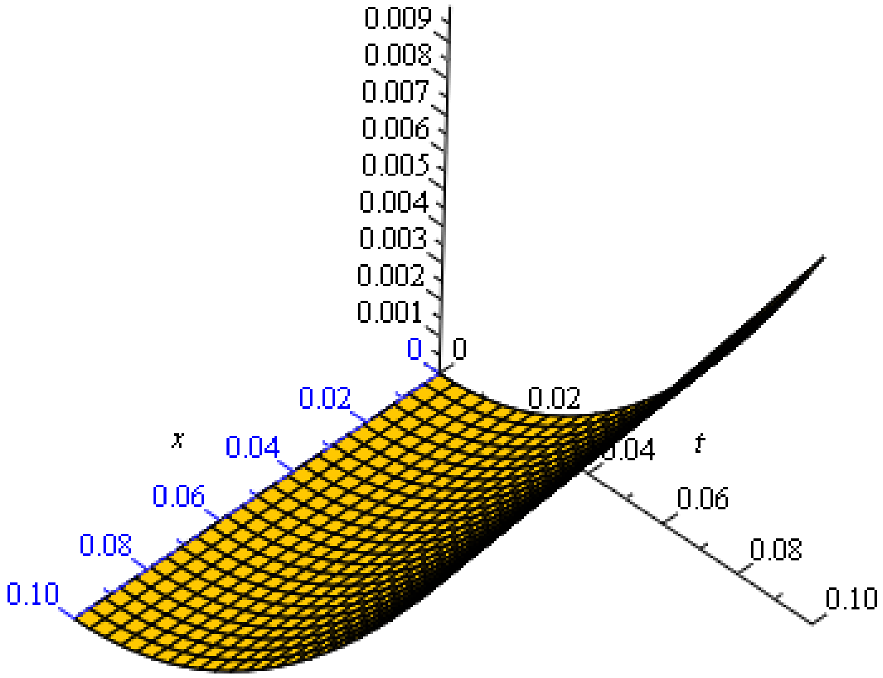
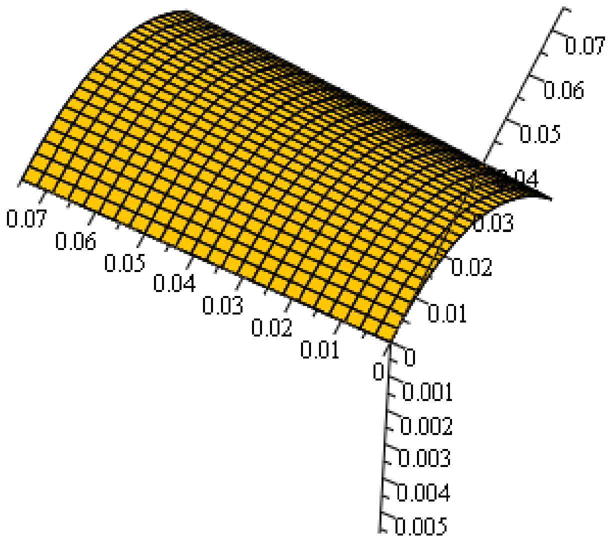
FIGURES

Figure:1 Sketch of the Lagrangian Formulation of Equation (5.4)



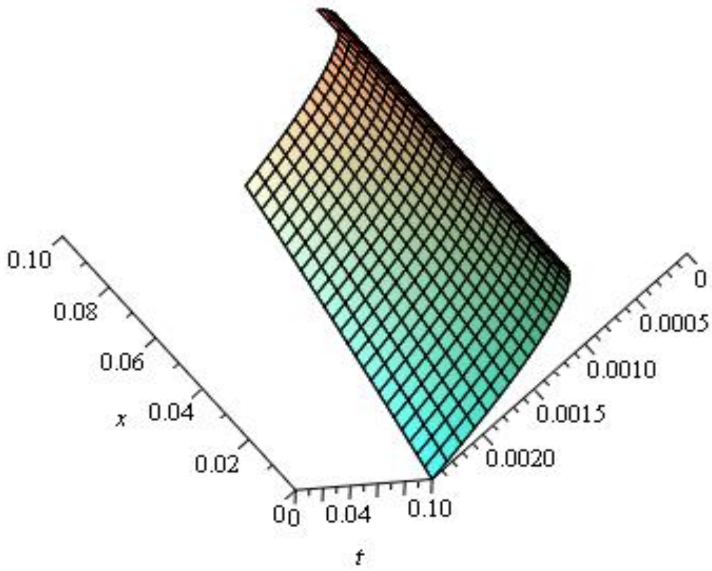
For A=1,m=1 and ω=1

Figure: 2 Sketch of the Lagrangian Formulation of Equation (5.4)



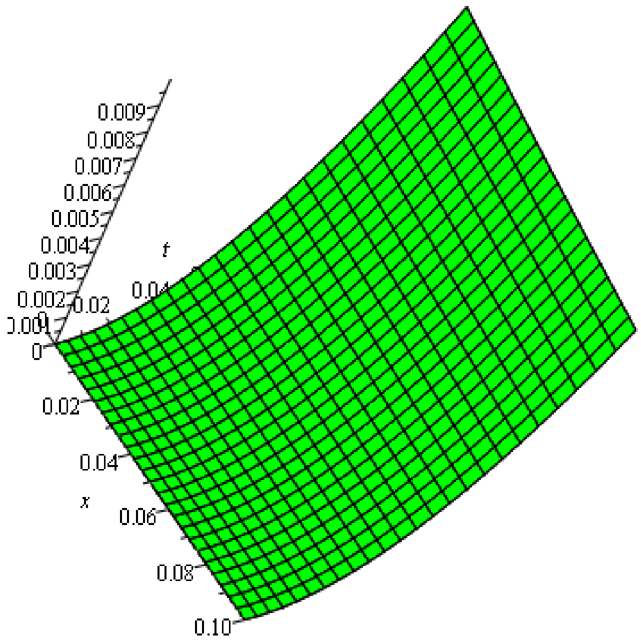
A=1,m=1,ω=0.75

Figure:3 Sketch of the Lagrangian Formulation of Equation (5.4)



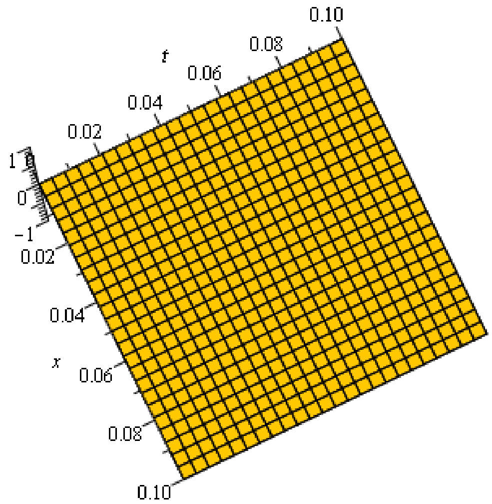
For A=1, m=1,ω=0.50

Figure:4 Sketch of the Lagrangian Formulation of Equation (5.4)



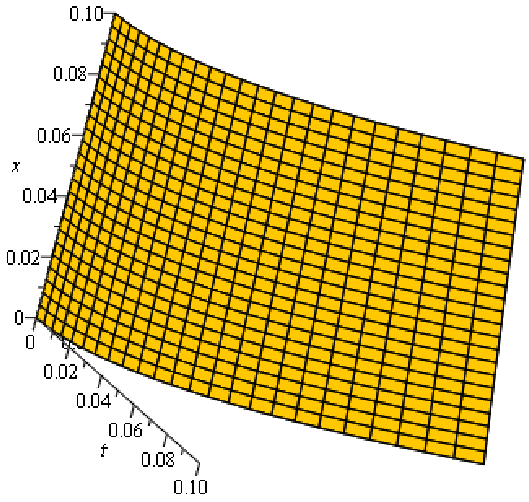
A=1,m=1 and ω=0.25

Figure:5 Sketch of the Lagrangian Formulation of Equation (5.4)



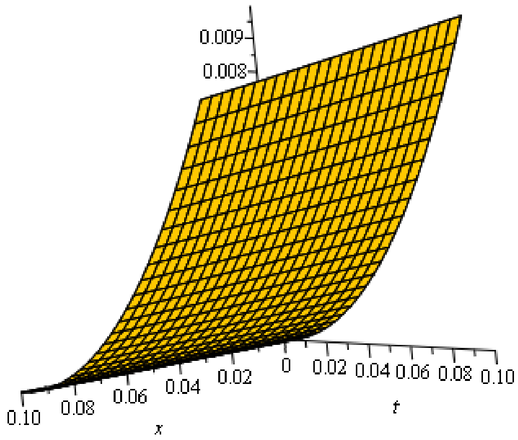
For A=1, m=1 and ω=0

Figure:6 Sketch of the Lagrangian Formulation of Equation (5.4)



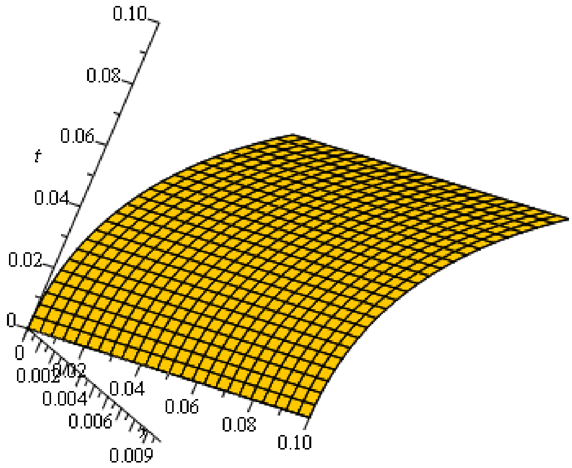
For A=1,m=0.75 and ω=1

Figure:7 Sketch of the Lagrangian Formulation of Equation (5.4)



For A=1,m=0.50 and ω=1

Figure: 8 Sketch of the Lagrangian Formulation of Equation (5.4)



For A=1,m=0.25 and ω=1